Pergamon

PII: S0021-8928(96)00083-4

J. Appl. Maths Mechs, Vol. 60, No. 4, pp. 665–675, 1996

0021-8928/96 $24.00+0.00

A DELAMINATED INCLUSION IN THE CASE OF ADHESION AND SLIPPAGE†

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(*Received* 20 *April* 1995)

The plane problem of stress concentration near a thin absolutely rigid inclusion is considered. Under the action of a force and a moment, applied to the upper edge of the inclusion, which is completely bonded to an elastic medium, the lower edge of the inclusion separates into layers: a crack opens in a certain inner section and finite slippage zones occur outside it. The problem is equivalent to a system of four singular integral equations in different sections. In the symmetric case, the reduction of this system to a single singular integral equation of the Mellin-convolution type in the interval (μ, 1) turns out to be effective, as the latter equation can be solved using a previously proposed scheme [1] as a consequence of the smallness of μ. In the general case, the system is reduced to two Riemann vector problems which are solved successively and for which analytic and asymptotic solutions are constructed. The zones of slippage and detachment, the angle of rotation of the inclusion, the normal displacements of the lower edge of the inclusion and the contact stresses in the slippage zone are found. Copyright © 1996 Elsevier Science Ltd.

A fundamental mixed problem for a crack, that is, the problem of the separation of a layer from an inclusion, has been solved earlier in [2–5] without introducing slippage zones in the neighbourhood of the inclusion ends. Even in the case of a homogeneous medium, this formulation leads to the non-physical oscillations back and forth of the edges of a crack in the neighbourhood of its vertices. A similar situation arises [6] in the problem of an interface crack and this was overcome in [7] by introducing slippage zones. An oscillating singularity in the case of stresses and displacements in the neighbourhood of the ends does not arise [8] in the antiplane problem for delaminated inclusion. An exact solution of the plane problem of contact between an inclusion and elastic material, when there are delamination sections and no account is taken of shear stresses, has been constructed in [9].

1. DELAMINATION UNDER THE ACTION OF A CENTRALLY APPLIED VERTICAL FORCE

There is an absolutely rigid inclusion ($-a \leqslant x \leqslant a, y = \pm 0$) in a homogeneous elastic plane and a vertical force P is applied at the point $x = 0$ to the upper edge of the inclusion which is bonded to the medium

$$u = 0, \quad v = 0, \quad -a \leqslant x \leqslant a, \quad y = +0 \tag{1.1}$$

The lower edge of the inclusion peels off. The median segment ($-b$, b) of the crack ($-a < x < a$, $y = -0$) opened under the action of the force P

$$\sigma_y = 0, \quad \tau_{xy} = 0, \quad -b < x < b, \quad y = -0 \tag{1.2}$$

and the conditions of frictionless slippage

$$\tau_{xy} = 0, \quad v = 0, \quad b < |x| < a, \quad y = -0 \tag{1.3}$$

are satisfied in the end sections ($-a$, $-b$), (b, a).

The position of point b is determined when solving the problem from the condition for the smooth joining of the lower edge of the crack

$$\partial v/\partial x(x, -0) \to 0, \quad x \to \pm b \mp 0$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 4, pp. 669–680, 1996.

It follows that one should accept as being physically correct that solution which eliminates tensile contact stresses in the slippage zone and the oscillations back and forth of crack edges in the separation zone

$$\sigma_y(x,-0) \leqslant 0,\quad b<|x|<a;\quad v(x,-0)\leqslant 0,\quad -b<x<b \tag{1.4}$$

We introduce the discontinuities

$$\chi_1(x)=\langle\sigma_y\rangle,\quad \chi_2(x)=\langle\tau_{xy}\rangle,\quad \chi_3(x)=2G\left\langle\frac{\partial u}{\partial x}\right\rangle,\quad \chi_4(x)=2G\left\langle\frac{\partial v}{\partial x}\right\rangle$$

$$(\langle f\rangle = f|_{y=-0} - f|_{y=+0})$$

and express the contact stresses and the tangential derivatives of the displacements on the line $y = 0$ in terms of them

$$\begin{aligned}
&2\varkappa^+\sigma_y(x,-0)=\varkappa^+\chi_1(x)-\varkappa^-\Gamma_a\chi_2(x)-\Gamma_b\chi_4(x)\\
&2\varkappa^+\tau_{xy}(x,-0)=\varkappa^-\Gamma_a\chi_1(x)+\varkappa^+\chi_2(x)-\Gamma_a\chi_3(x)\\
&4G\varkappa^+\partial u/\partial x(x,+0)=\varkappa\Gamma_a\chi_2(x)-\varkappa^+\chi_3(x)-\varkappa^-\Gamma_b\chi_4(x)\\
&4G\varkappa^+\partial v/\partial x(x,+0)=\varkappa\Gamma_a\chi_1(x)+\varkappa^-\Gamma_a\chi_3(x)-\varkappa^+\chi_4(x)\\
&\varkappa=3-4\nu,\quad \varkappa^\pm=\frac{\varkappa\pm 1}{2},\quad \Gamma_c\chi(x)=\frac{1}{\pi}\int_{-c}^{c}\frac{\chi(\xi)}{\xi-x}d\xi
\end{aligned}\tag{1.5}$$

On satisfying conditions (1.1)–(1.3), we arrive at the system of four singular integral equations

$$\varkappa^+\chi_1(x)-\varkappa^-\Gamma_a\chi_2(x)-\Gamma_b\chi_4(x)=0,\quad -b<x<b \tag{1.6}$$

$$\varkappa^-\Gamma_a\chi_1(x)+\varkappa^+\chi_2(x)-\Gamma_a\chi_3(x)=0,\quad -a<x<a \tag{1.7}$$

$$\varkappa\Gamma_a\chi_2(x)-\varkappa^+\chi_3(x)-\varkappa^-\Gamma_b\chi_4(x)=0,\quad -a<x<a \tag{1.8}$$

$$\varkappa\Gamma_a\chi_1(x)+\varkappa^-\Gamma_a\chi_3(x)-\varkappa^+\chi_4(x)=0,\quad -a<x<a \tag{1.9}$$

with the supplementary conditions for the closure of the slit and the equilibrium of the inclusion

$$\int_{-a}^{a}\chi_3(x)dx=0,\quad \int_{-a}^{a}\chi_1(x)dx=P \tag{1.10}$$

(the remaining conditions are automatically satisfied by virtue of the evenness of $\chi_1(x)$ and the oddness of $\chi_2(x)$, $\chi_4(x)$). On expressing the functions $\Gamma_a\chi_1(x)$ and $\Gamma_a\chi_3(x)$ from Eqs (1.7) and (1.9) and applying the operator Γ_a in a class of functions with integrable singularities at the ends $x = \pm a$, we find

$$\begin{aligned}
\chi_1(x)&=-\frac{1}{(a^2-x^2)^{1/2}}\left[C_0+\frac{1}{\pi}\int_{-a}^{a}\frac{(a^2-\xi^2)^{1/2}}{\xi-x}\left(-\frac{\varkappa^-}{\varkappa^+}\chi_2(\xi)+\frac{1}{\varkappa^+}\chi_4(\xi)\right)d\xi\right]\\
\chi_3(x)&=-\frac{1}{(a^2-x^2)^{1/2}}\left[C_1+\frac{1}{\pi}\int_{-a}^{a}\frac{(a^2-\xi^2)^{1/2}}{\xi-x}\left(\frac{\varkappa}{\varkappa^+}\chi_2(\xi)+\frac{\varkappa^-}{\varkappa^+}\chi_4(\xi)\right)d\xi\right]
\end{aligned}\tag{1.11}$$

On satisfying the supplementary conditions (1.10), we have $C_1 = 0$, $C_0 = -P\pi^{-1}$ and, on substituting expressions (1.11) into Eqs (1.6) and (1.8) and taking into account the oddness of the functions χ_2 and χ_4, after the change of variables $\xi = a\xi_1$, $x = ax_1$ we obtain

$$\frac{\varkappa^-}{\pi}\int_0^1\left[-1+\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2}\right]\frac{2\xi\chi_2(a\xi)}{\xi^2-x^2}d\xi-\frac{1}{\pi}\int_0^{\lambda_0}\left[1+\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2}\right]\frac{2\xi\chi_4(a\xi)}{\xi^2-x^2}d\xi=-\frac{\varkappa^+P}{\pi a(1-x^2)^{1/2}}$$

$$0 < x < \lambda_0, \quad \lambda_0 = b/a \tag{1.12}$$

$$\frac{\varkappa}{\pi}\int_0^1\left[1+\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2}\right]\frac{2\xi\chi_2(a\xi)}{\xi^2-x^2}d\xi+\frac{\varkappa^-}{\pi}\int_0^{\lambda_0}\left[-1+\left(\frac{1-\xi^2}{1-x^2}\right)^{1/2}\right]\frac{2\xi\chi_4(a\xi)}{\xi^2-x^2}d\xi=0$$

$$0 < x < 1$$

Analysis of the Cauchy-type integrals in (1.6)–(1.9), as well as the condition for the smoothness of the profile of the slit at the points $x = \pm b$, leads to the following class of solutions

$$\chi_2(\xi)=O(\xi),\ \ \xi\to+0;\quad \chi_2(\xi)=O\{(a-\xi)^{-3/4}\},\ \ \xi\to a-0$$

$$\chi_4(\xi)=O(\xi),\ \ \xi\to+0;\quad \chi_4(\xi)=O\{(b-\xi)^{1/2}\},\ \ \xi\to b-0$$

We now introduce the new functions

$$\varphi_1(\eta)=2\eta\chi_2(a(1-\eta^2)^{1/2}),\quad \varphi_2(\eta)=2\eta\chi_4(a(1-\eta^2)^{1/2}) \tag{1.13}$$

and simplify system (1.12) to the form

$$-\frac{\varkappa^-}{\pi}\int_0^1\frac{\varphi_1(\eta)}{\eta+y}d\eta+\frac{1}{\pi}\int_\lambda^1\frac{\varphi_2(\eta)}{\eta-y}d\eta=-\frac{P\varkappa^+}{\pi a},\quad \lambda<y<1$$

$$\frac{1}{\pi}\int_0^1\frac{\varphi_1(\eta)}{\eta-y}d\eta+\frac{\varkappa^-}{\pi\varkappa}\int_\lambda^1\frac{\varphi_2(\eta)}{\eta+y}d\eta=0,\quad 0<y<1 \tag{1.14}$$

where $\lambda = (1-\lambda_0^2)^{1/2}$ and the functions φ_1, φ_2 have the asymptotic forms

$$\varphi_1(\eta)=O\{(1-\eta)^{1/2}\},\ \ \eta\to1-0;\quad \varphi_1(\eta)=O(\eta^{-1/2}),\ \ \eta\to+0 \tag{1.15}$$

$$\varphi_2(\eta)=O\{(1-\eta)^{1/2}\},\ \ \eta\to1-0;\quad \varphi_2(\eta)=O\{(\eta-\lambda)^{1/2}\},\ \ \eta\to\lambda+0$$

We express the function $\varphi_1(\eta)$ from the second equation of (1.14), taking account of (1.15), as

$$\varphi_1(\eta)=\frac{\varkappa^-}{\pi\varkappa}\left(\frac{1-\eta}{\eta}\right)^{1/2}\int_\lambda^1\left(\frac{\xi}{1+\xi}\right)^{1/2}\frac{\varphi_2(\xi)}{\xi+\eta}d\xi \tag{1.16}$$

This last relation is substituted into the first equation of (1.14) and use is made of the equality

$$\frac{1}{\pi}\int_0^1\frac{\xi^{-1/2}(1-\xi)^{1/2}d\xi}{(\xi+y)(\xi+\eta)}=\frac{2^{1/2}}{\eta-y}\left(\frac{1}{t}-\frac{1}{\tau}\right),\quad \tau=\left(\frac{2\eta}{1+\eta}\right)^{1/2},\quad t=\left(\frac{2y}{1+y}\right)^{1/2},\quad 0<\eta,\ y<1 \tag{1.17}$$

As a result, we arrive at an integral equation which, after introducing the variables τ, t and the functions

$$\varphi(\tau)=-\frac{\pi\varkappa^+a}{P\varkappa(2-\tau^2)}\varphi_2\left(\frac{\tau^2}{2-\tau^2}\right) \tag{1.18}$$

reduces to an equation of the Mellin-convolution type

$$\frac{2}{\pi}\int_\mu^1\frac{\varphi(\tau)\tau}{t^2-\tau^2}\left(1-\gamma^2\frac{\tau}{t}\right)d\tau=\frac{1}{t^2-2},\quad \mu<t<1,\quad \mu=\left(\frac{2\lambda}{1+\lambda}\right)^{1/2},\quad \gamma=\frac{\varkappa^-}{\varkappa^+} \tag{1.19}$$

We extend this equation along the positive half-axis

$$\frac{2}{\pi}\int_0^\infty \varphi_*(\tau)\frac{1-\gamma^2\tau/t}{(t/\tau)^2-1}\frac{d\tau}{\tau}=f_*(t)+\varphi_-(t)+\varphi_+(t),\quad 0<t<\infty \tag{1.20}$$

$$f_*(t)=\begin{cases}(t^2-2)^{-1}, & 0<t<1\\ 0, & t>1\end{cases},\quad \varphi_*(t)=\begin{cases}\varphi(t), & \mu<t<1\\ 0, & t\,\overline{\in}\,(\mu,1)\end{cases}$$

$$\operatorname{supp}\varphi_-(t)\subset[0,\mu],\quad \operatorname{supp}\varphi_+(t)\subset[1,\infty)$$

and introduce the Mellin transforms

$$\Phi_1^-(s)=\int_\mu^1\varphi(\tau)\tau^s d\tau,\quad \Phi_1^+(s)=\int_1^{1/\mu}\varphi(\mu\tau)\tau^s d\tau$$

$$\Phi_2^-(s)=\int_0^1\varphi_-(\mu\tau)\tau^s d\tau,\quad \Phi_2^+(s)=\int_1^\infty\varphi_+(\tau)\tau^s d\tau$$

$$F^-(s)=\int_0^1\frac{\tau^s}{\tau^2-2}d\tau=-\sum_{j=0}^\infty\frac{1}{2^{j+1}(s+2j+1)},\quad \operatorname{Re}(s)>-1$$

The application of a Mellin transform to Eq. (1.20) leads to the following Riemann vector problem

$$\Phi_1^+(s)=\mu^{-s-1}\Phi_1^-(s)$$

$$\Phi_2^+(s)=(\operatorname{tg}\tfrac{1}{2}\pi s+\gamma^2\operatorname{ctg}\tfrac{1}{2}\pi s)\Phi_1^+(s)-\mu^{-s-1}\Phi_2^-(s)-F^-(s)$$

$$s\in\Gamma:\ \operatorname{Re}(s)=\gamma_0\in(0,1)$$

The solution of this problem is constructed using a scheme which has been described previously [1]. We shall only present the final formulae

$$\Phi_1^-(s)=[K^-(s)]^{-1}\Sigma(s)+\mu^{s+1}[K^+(s)]^{-1}\Psi^-(s),\quad \Phi_2^-(s)=K^-(s)\Psi^-(s) \tag{1.21}$$

$$\Phi_2^+(s)=K^+(s)\Sigma(s)+\sum_{j=0}^\infty\frac{2^{-j-1}}{s+2j+1},\quad \Sigma(s)=\Psi^+(s)-\sum_{j=0}^\infty\frac{2^{-j-1}}{K^+(-2j-1)(s+2j+1)}$$

$$K^+(s)=\frac{(1-\gamma^2)\Gamma(1-s/2)\Gamma(\tfrac{1}{2}-s/2)}{\Gamma(1-s/2-i\beta/2)\Gamma(1-s/2+i\beta/2)},\quad K^-(s)=\frac{\Gamma(s/2)\Gamma(\tfrac{1}{2}+s/2)}{\Gamma(s/2+i\beta/2)\Gamma(s/2-i\beta/2)}$$

$$\Psi^+(s)=\sum_{j=1}^\infty\frac{A_j^+}{s-s_j},\quad \Psi^-(s)=\sum_{j=1}^\infty\frac{A_j^-}{s+s_j-2},\quad \beta=\frac{1}{\pi}\ln\frac{1+\gamma}{1-\gamma}=\frac{\ln\varkappa}{2\pi} \tag{1.22}$$

$$s_{2j-1}=i\beta+2j,\quad s_{2j}=-i\beta+2j$$

The coefficients $A_j^\pm$ are the solution of the infinite algebraic system

$$A_m^+=\mu^{s_m+1}\Delta_m^-\sum_{j=1}^\infty\frac{A_j^-}{s_m+s_j-2} \tag{1.23}$$

$$A_m^-=\mu^{s_m-3}\Delta_m^+\left(f_m+\sum_{j=1}^\infty\frac{A_j^+}{2-s_m-s_j}\right)\quad (m=1,2\ldots)$$

$$\Delta_{2m-1}^+=-\frac{1-\gamma^2}{\pi}\left[\frac{\Gamma(m+i\beta/2)\Gamma(m-\tfrac{1}{2}+i\beta/2)}{\Gamma(m)\Gamma(m+i\beta)}\right]^2,\quad \Delta_{2m}^+=\overline{\Delta_{2m-1}^+}$$

$$\Delta_{2m-1}^-=\frac{(m-\tfrac{1}{2}+i\beta/2)^2}{(1-\gamma^2)^2}\Delta_{2m-1}^+,\quad \Delta_{2m}^-=\overline{\Delta_{2m-1}^-}$$

$$f_m = \frac{1}{1-\gamma^2}\sum_{j=0}^{\infty}\frac{|\Gamma(3/2+i\beta/2+j)|^2}{2^{j+1}\Gamma(3/2+j)j!(s_m-3-2j)}$$

In order that $\chi_4(\xi) = O\{(b-\xi)^{1/2}\}, \xi \to b-0$ or, what is equivalent, $\varphi(\tau) = O\{(\tau-\mu)^{1/2}\}, \tau \to \mu + 0$, it is necessary and sufficient that the quantity μ should be a root of the following equation

$$\sum_{j=0}^{\infty} A_j^- = 0 \tag{1.24}$$

Then

$$b = 2a(2-\mu^2)^{-1}(1-\mu^2)^{1/2}$$

A numerical analysis of the solution of system (1.23) shows that, as in the problem of an interface crack [1], Eq. (1.24) can only have a root among the values of μ which are close to zero. In order to analyse the solution of the system for small μ, we transform system (1.23) in terms of the recurrence relations

$$A_m^{\pm} = \mu^{s_m - 3/2 \pm 1/2}\sum_{k=1}^{\infty} a_{mk}^{\pm}\mu^{2k-2}, \quad a_{m1}^{-} = f_m\Delta_m^{+} \tag{1.25}$$

$$a_{mn\pm}^{\pm} = \pm\Delta_m^{\mp}\sum_{j=1}^{n}\left(\frac{\mu^{i\beta}a_{2j-1,n+1-j}^{\mp}}{s_m+s_{2j-1}-2}+\frac{\mu^{-i\beta}a_{2j,n+1-j}^{\mp}}{s_m+s_{2j}-2}\right)$$

$$n_+ = n, \quad n_- = n+1; \quad n = 1,2,\ldots; \quad m = 1,2,\ldots$$

On substituting formulae (1.25) into Eq. (1.24), we obtain

$$\mu^{2i\beta}a_{11}^- + a_{21}^- + O(\mu^2) = 0, \quad \mu \to 0$$

from which we find the following asymptotic formula

$$\mu_k = 4\exp\left\{-\frac{1}{2\beta}\operatorname{arctg}\frac{2q_1q_2}{q_1^2-q_2^2}-\frac{\pi}{\beta}\left(k+\frac{1}{2}\right)\right\}+O(\mu_k^2), \quad k = 0,1,2,\ldots$$

$$q_1 + iq_2 = (1+i\beta)F(3/2+i\beta/2,\ 1/2-i\beta/2;\ 3/2;\ 1/2), \quad \operatorname{Im}(q_1,q_2) = 0$$

The values of the first few roots for some values of ν are presented below

ν	10^{-6}	0.1	0.3	0.45
μ_0	0.212×10^{-1}	0.107×10^{-1}	0.417×10^{-3}	0.321×10^{-11}
μ_1	0.266×10^{-5}	0.350×10^{-6}	0.212×10^{-10}	0.993×10^{-35}
μ_2	0.333×10^{-9}	0.114×10^{-10}	0.108×10^{-17}	0.307×10^{-58}

It now remains to verify that the solution obtained is correct, that is, to verify the practicability of conditions (1.4). Let us determine the jump in the normal displacements $\langle v\rangle(x)$. By virtue of (1.13) and (1.18), we have

$$\chi_4(x) = \tfrac{1}{2}(1-x^2/a^2)^{-1/2}\varphi_2((1-x^2/a^2)^{1/2}) =$$

$$= \frac{-P\varkappa a}{\pi\varkappa^+(a^2-x^2)^{1/2}[a+(a^2-x^2)^{1/2}]}\varphi\left(\frac{2^{1/2}}{[1+(1-x^2/a^2)^{-1/2}]^{1/2}}\right)$$

where the function $\varphi(\tau)$, which is the solution of Eq. (1.19), is determined from (1.21) using an inverse Mellin transform and the theory of residues

$$\varphi(\tau)=\frac{1}{2\pi i}\int_{\Gamma}\Phi_1^-(s)\tau^{-s-1}ds=\sum_{m=1}^{\infty}\left(\frac{A_m^+\tau^{-s_m-1}}{K^-(s_m)}-\frac{A_m^-(\tau/\mu)^{s_m-3}}{K^+(2-s_m)}\right)$$

$$\mu<\tau<1 \tag{1.26}$$

We will now investigate the convergence of series (1.26) at the points $\tau=\mu$ and $\tau=1$. By virtue of (1.22), we have the asymptotic form when $s\to\infty$ $K^+(s)=O(s^{-1/2})$, $s\in D^+$, $K^-(s)=O(s^{1/2})$, $s\in D^-$, $D^\pm$: $\mathrm{Re}(s)\lessgtr\gamma_0$. When account is taken of equality (1.24), analysis of relations (1.23) gives

$$A_m^-=O(m^{-2}\mu^{s_m}),\quad A_m^+=O(m^{-1}\mu^{s_m}),\quad m\to\infty$$

so that the elements of series (1.26) behave as $m^{-3/2}$ when $m\to\infty$, and this means that

$$\varphi(\tau)=O\{(\tau-\mu)^{1/2}\},\quad \tau\to\mu+0;\quad \varphi(\tau)=O\{(1-\tau)^{1/2}\},\quad \tau\to1-0$$

The displacements of the lower peeling edge have the form

$$v(x,-0)=-\frac{1}{2G}\int_x^b\chi_4(\xi)d\xi$$

We now find the contact normal stresses $\sigma_y(x,-0)$ in the contact section $b<x<a$. Taking account of relation (1.5) and relationships (1.11), (1.13), (1.16), (1.18) and (1.20), we obtain

$$\frac{2\pi at^2}{P(2-t^2)^2}\sigma_y\left(\frac{2a(1-t^2)^{1/2}}{2-t^2},-0\right)=-\frac{1}{t^2-2}+\frac{2}{\pi}\int_\mu^1\frac{\tau\varphi(\tau)}{t^2-\tau^2}\left(1-\gamma^2\frac{\tau}{t}\right)d\tau=$$
$$=\varphi_-(t)+\varphi_+(t),\quad 0<t<\infty$$

On applying an inverse Mellin transform to the second equation of (1.21), we arrive at the relations

$$\sigma_y(x,-0)=\frac{Pa}{\pi A(x)[a+A(x)]}\varphi_-\left(\left[\frac{2A(x)}{a+A(x)}\right]^{1/2}\right),\quad b<x<a \tag{1.27}$$

$$A(x)=(a^2-x^2)^{1/2}$$

$$\varphi_-(t)=-2\sum_{m=1}^{\infty}\frac{(-1)^m}{\Gamma(m)}\left(\frac{t}{\mu}\right)^{2m-2}\left[\frac{\Gamma(3/2-m)\Psi^-(2-2m)}{|\Gamma(1-m+i\beta/2)|^2}\frac{\mu}{t}+\frac{\Gamma(1/2-m)\Psi^-(1-2m)}{|\Gamma(1/2-m+i\beta/2)|^2}\right]$$

Analysis of these formulae shows that the contact stresses increase monotonically in the neighbourhood of the point $x=a$ and, moreover, that

$$\sigma_y(x,-0)=O\{(a-x)^{-3/4}\},\quad x\to a-0;\quad \sigma_y(x,-0)=O\{(x-b)^{1/2}\},\quad x\to b+0$$

The calculations lead to the conclusion that the stresses and displacements satisfy only conditions (1.4) in the case of the root μ_0 (the largest among the roots of (1.24)).

The results of the calculation presented below

ν	0	0.1	0.2	0.3	0.4	0.45
μ	0.212×10^{-1}	0.107×10^{-1}	0.356×10^{-2}	0.417×10^{-3}	0.782×10^{-6}	0.321×10^{-11}
λ	0.225×10^{-3}	0.575×10^{-4}	0.633×10^{-5}	0.867×10^{-7}	0.306×10^{-12}	0.515×10^{-23}
l/a	0.252×10^{-7}	0.165×10^{-8}	0.201×10^{-10}	0.376×10^{-14}	0.468×10^{-25}	0.133×10^{-46}

show that the length of the contact $l=a-b$ is a maximum when $\nu=0$ and decreases to zero as Poisson's ratio increases up to $\nu=0.5$.

A graph of the displacements $-2Gv(x,-0)/P$ in the separation zone $0<x<b$ for $\nu=0.3$ is shown in Fig. 1. Note that the derivative with respect to x of the function $v(x,-0)$ vanishes at the point $x=b$. Plots of the dependence of the function $\varphi_-^*(\tau)=-10^{-2}\varphi_-(\tau)$ on $\tau=10^{-2}\tau_0$ for the case when $\nu=0.1$ (curve 1) and of the function $\varphi_-^*(\tau)=-10^{-3}\varphi_-(\tau)$ on $\tau=10^{-3}\tau_0$ for $\nu=0.3$ (curve 2) are also shown.

In both cases $0 \leqslant \tau \leqslant \mu$. The normal stresses are connected with the function $\varphi_-(\tau)$ by relation (1.27). It can be seen from the graphs in Fig. 1 that the solution satisfies the correctness conditions (1.4). Compressive contact stresses arise in the small segments where slippage occurs (with a length of less than $2.5 \times 10^{-8}a$). These stresses are equal to zero at the point $x = b$ and tend to infinity monotonically as $x \to a - 0$. The stresses $x \to a + 0$ are bounded when $\sigma_y(x, 0)$. This fact follows from an analysis of the behaviour of the function $\Phi^+{}_2(s)$ when $s \to \infty$, $s \in D^+$ and theorems of the Abel type.

2. AN INCLUSION IN THE CASE OF ASYMMETRIC LOADING

Let a vertical force P, a horizontal force T and a moment M be applied to an absolutely rigid inclusion $(0 \leqslant x \leqslant 1, y = \pm 0)$ at the point $x = 1/2, y = +0$ (Fig. 2). The upper edge of the inclusion is completely bonded while the lower edge peels off as a layer and, moreover, the segment which has peeled off is subdivided into three intervals: two slippage zones $(0, b_1)$ and (b_2, a) and a single separation zone (b_1, b_2)

$$\begin{aligned} &u(x,+0) = 0, \quad v(x,+0) = \gamma x, \quad 0 < x < 1 \\ &\sigma_y(x,-0) = 0, \quad b_1 < x < b_2; \quad \tau_{xy}(x,-0) = 0, \quad 0 < x < 1 \\ &v(x,-0) = \gamma x, \quad 0 < x < b_1, \quad b_2 < x < a \end{aligned} \tag{2.1}$$

where γ is the turn angle of the inclusion.

Using representations (1.5) and boundary conditions (2.1), we arrive at the system of singular integral equations

$$\varkappa^+\chi_1(x) - \varkappa^-\Gamma\chi_2(x) - \Gamma\chi_4(x) = 0, \quad b_1 < x < b_2 \tag{2.2}$$

$$\varkappa^-\Gamma\chi_1(x) + \varkappa^+\chi_2(x) - \Gamma\chi_3(x) = 0, \quad 0 < x < 1 \tag{2.3}$$

$$\varkappa\Gamma\chi_2(x) - \varkappa^+\chi_3(x) - \varkappa^-\Gamma\chi_4(x) = 0, \quad 0 < x < 1 \tag{2.4}$$

$$\varkappa\Gamma\chi_1(x) + \varkappa^-\Gamma\chi_3(x) - \varkappa^+\chi_4(x) = 4G\varkappa^+\gamma, \quad 0 < x < 1 \tag{2.5}$$

$$\Gamma\chi(x) = \frac{1}{\pi}\int_0^1 \frac{\chi(\xi)}{\xi - x}d\xi, \quad \operatorname{supp}\chi_4(x) \subset [b_1, b_2]$$

The solution of this system completely satisfies the following supplementary conditions for the closure of the slit and the equilibrium of the inclusion

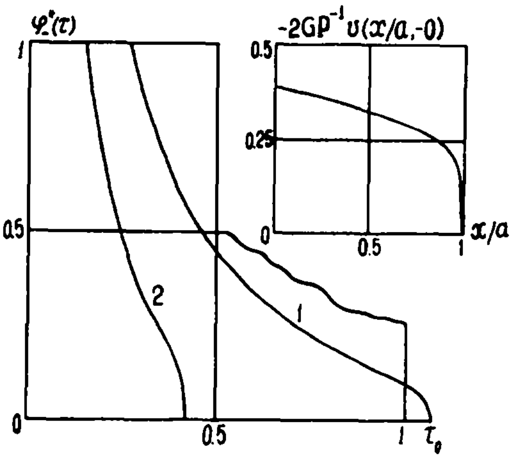

Fig. 1.

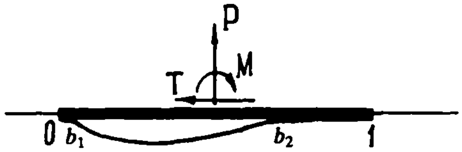

Fig. 2.

$$\int_0^1 \chi_3(x)dx = 0, \quad \int_{b_1}^{b_2} \chi_4(x)dx = 0,$$

$$\int_0^1 \chi_1(x)dx = P, \quad \int_0^1 \chi_2(x)dx = -T, \quad \int_0^1 \chi_1(x)xdx = M \tag{2.6}$$

Without loss of generality, we will assume that the left-hand zone where slippage occurs is smaller, that is, $b_1 \ll b_2$. The points b_1 and b_2 are determined when solving the problem from the smoothness conditions

$$\chi_4(x) \to 0, \quad x \to b_1 + 0; \quad \chi_4(x) \to 0, \quad x \to b_2 - 0$$

We now express $\Gamma\chi_1(x)$ from Eq. (2.5), substitute into (2.3) and then introduce the functions $\varphi_2(x)$, $\varphi_3(x)$

$$\chi_2(x) = \varphi_2(x) - (\varkappa^+/\varkappa)\varphi_3(x), \quad \chi_3(x) = -(\varkappa/\varkappa^+)\varphi_2(x) - \varphi_3(x) \tag{2.7}$$

Equations (2.3) and (2.4) then become

$$(I+\Gamma)\varphi_2(x) + (2\varkappa)^{-1}\varkappa^-(I-\Gamma)\chi_4(x) = -\gamma_*, \quad 0 < x < 1$$

$$(I-\Gamma)\varphi_3(x) - (2\varkappa^+)^{-1}\varkappa^-(I+\Gamma)\chi_4(x) = (\varkappa^+)^{-1}\varkappa\gamma_*, \quad 0 < x < 1$$

$$\gamma_* = 2G(\varkappa^-/\varkappa)\gamma$$

where I is the identity operator. We extend the definition of the last two equations to values of $x > 1$ using the functions $\varphi_{2+}(x)$ and $\varphi_{3+}(x)$, respectively, and apply a Mellin transform. We now have

$$c_+(s)\Phi_2^-(s) + \varkappa^-(2\varkappa)^{-1}b_2^{s+1}c_-(s)\Phi_4^-(s) = -\gamma_*(s+1)^{-1} + \Phi_2^+(s)$$

$$c_-(s)\Phi_3^-(s) - \varkappa^-(2\varkappa^+)^{-1}b_2^{s+1}c_+(s)\Phi_4^-(s) = \varkappa[\varkappa^+(s+1)]^{-1}\gamma_* + \Phi_3^+(s), \quad -1/4 < \mathrm{Re}(s) < 0 \tag{2.8}$$

$$\Phi_m^-(s) = \int_0^1 \varphi_m(x)x^s dx, \quad \Phi_m^+(s) = \int_1^\infty \varphi_{m+}(x)x^s dx \quad (m = 2, 3)$$

$$\Phi_4^-(s) = \int_{b_1/b_2}^1 \chi_4(b_2x)x^s dx, \quad c_\pm(s) = \mathrm{ctg}\,\pi s \pm 1$$

We factorize the functions $c_\pm(s)$ and construct a solution of problem (2.8) without using Cauchy-type integrals and assuming that the function $\Phi_4^-(s)$ is temporarily known

$$\Phi_2^+(s) = K_0^+(s)\Sigma_1(s) + \frac{\gamma_*}{s+1}, \quad \Phi_2^-(s) = \frac{\Sigma_1(s)}{K_0^-(s)} - \frac{\varkappa^- b_2^{s+1}\Phi_4^-(s)}{2\varkappa\,\mathrm{ctg}\,\pi(s-1/4)} \tag{2.9}$$

$$\Phi_3^+(s) = K_1^+(s)\Sigma_2(s) - \frac{\varkappa\gamma_*}{\varkappa^+(s+1)}, \quad \Phi_3^-(s) = \frac{\Sigma_2(s)}{K_1^-(s)} + \frac{\varkappa^- b_2^{s+1}\Phi_4^-(s)}{2\varkappa^+\,\mathrm{tg}\,\pi(s-1/4)}$$

$$\Sigma_m(s) = C_m + \Psi_m^+(s) + \frac{(-1)^m\gamma_* e_m}{s+1}, \quad \Psi_m^+(s) = \sum_{k=1}^\infty \frac{A_{mk}}{s-k+m/2-1/4} \quad (m = 1, 2)$$

$$K_n^+(s) = (-1)^{n-1} 2^{1/2} \Gamma(-s)[\Gamma(3/4 - n/2 - s)]^{-1}, \quad e_1 = [K_0^+(-1)]^{-1}$$
$$K_n^-(s) = \Gamma(1+s)[\Gamma(1/4 + n/2 + s)]^{-1} \quad (n = 0, 1); \quad e_2 = \varkappa[\varkappa^+ K_1^+(-1)]^{-1}$$

where A_{mk} are coefficients which are subsequently to be determined from the conditions that the functions $\Phi_2^-(s)$ and $\Phi_3^-(s)$ should be analytic at the points $s = k - 1/4$ and $s = k - 3/4$ ($k = 1, 2, \ldots$), respectively, and C_1 and C_2 are arbitrary constants.

Taking account of (2.7), we now transform the remaining two equations of (2.2) and (2.5) to the form

$$\begin{aligned}
&\varkappa^+\chi_{1-}(x) - \varkappa^-\Gamma_*\varphi_{2-}(x) + \varkappa^+\varkappa^- / \varkappa\Gamma_*\varphi_{3-}(x) - \Gamma_*\chi_{4-}(x) = \chi_{0-}(x) + \chi_{0+}(x) \\
&\Gamma_*\chi_{1-}(x) - \varkappa^- / \varkappa^+\Gamma_*\varphi_{2-}(x) - \varkappa^- / \varkappa\Gamma_*\varphi_{3-}(x) - \varkappa^+ / \varkappa\chi_{4-}(x) = \\
&= 2(\varkappa^+ / \varkappa^-)\gamma_*\gamma_-(x) + \chi_{1+}(x) \\
&0 < x < \infty \\
&\operatorname{supp}\chi_{0-}(x) \subset [0, b_1], \quad \operatorname{supp}\chi_{0+}(x) \subset [b_2, \infty), \quad \operatorname{supp}\chi_{1+}(x) \subset [1, \infty) \\
&\|\chi_{1-}(x),\ \varphi_{2-}(x),\ \varphi_{3-}(x),\ \gamma_-(x)\| = \begin{cases} \|\chi_1, \varphi_2, \varphi_3, 1\|, & 0 < x < 1 \\ \|0, 0, 0, 0\|, & x > 1 \end{cases} \\
&\Gamma_*\chi(x) = \frac{1}{\pi}\int_0^\infty \frac{\chi(\xi)d\xi}{\xi - x}
\end{aligned} \tag{2.10}$$

On denoting the Mellin transforms with a weight x^s of the functions $\chi_{0-}(b_1, x)$, $\chi_{1-}(x)$, $\chi_{0+}(b_2x)$ and $\chi_{1+}(x)$ by $\Phi_0^-(s)$, $\Phi_1^-(s)$, $\Phi_0^+(s)$ and $\Phi_1^+(s)$, respectively, and taking account of relation (2.9), the system of integral equations (2.10) is reduced to the three functional equations

$$\begin{aligned}
&\Phi_1^-(s) - \varkappa^-[\varkappa^+ K_0^-(s)]^{-1}\Sigma_1(s) - \varkappa^-[\varkappa K_1^-(s)]^{-1}\Sigma_2(s) + b_2^{s+1}h(s)\Phi_4^-(s) = \\
&= \operatorname{tg}\pi s\{\Phi_1^+(s) + 2\varkappa^+\gamma_*[\varkappa^-(s+1)]^{-1}\} \\
&g(s)\Phi_4^-(s) + (b_1/b_2)^{s+1}\Phi_0^-(s) = -\Phi_0^+(s) + b_2^{-s-1}\Omega(s) \\
&\Phi_0^-(s) = -(b_1/b_2)^{-s-1}[g(s)\Phi_4^-(s) + \Phi_0^+(s)] + b_1^{-s-1}\Omega(s) \\
&\Omega(s) = \varkappa^+ \operatorname{tg}\pi s\left[\Phi_1^+(s) + \frac{2\varkappa^+\gamma_*}{\varkappa^-(s+1)}\right] - \frac{\varkappa^- c_-(s)}{K_0^-(s)}\Sigma_1(s) + \frac{\varkappa^+\varkappa^- c_+(s)}{\varkappa K_1^-(s)}\Sigma_2(s) \\
&h(s) = \frac{\varkappa^-}{\varkappa}\left(\frac{\varkappa^-}{\varkappa^+}\operatorname{tg}2\pi s - \frac{\varkappa^+}{\varkappa^-}\operatorname{tg}\pi s\right), \quad g(s) = \frac{2\varkappa\cos 4\pi s + \varkappa^2 + 1}{\varkappa\sin 4\pi s}
\end{aligned} \tag{2.11}$$

A solution of the Riemann vector equation (2.11) is constructed using the scheme which has been proposed earlier [1]. Its solution is determined apart from an arbitrary constant C_3. The constants C_1, C_2 and C_3 are found from the first, third and fourth conditions of (2.6). From the remaining two conditions we obtain the following formula for the turn angle of the inclusion $\gamma = (2G\varkappa^-)^{-1}\varkappa\gamma_*$

$$\begin{aligned}
\gamma_* = &\left\{\sum_{k=1}^\infty \left[\frac{\varkappa^-\Gamma_1 A_{1k}}{\varkappa^+(k - 1/4)(k - 5/4)} + \frac{3\varkappa^-\Gamma_0 A_{2k}}{\varkappa(k - 3/4)(k - 7/4)} + \frac{2\pi^{1/2} A_{3k}}{(k - 1/2)(k - 3/2)}\right] - \right. \\
&\left. -2P - (\varkappa^-/\varkappa^+)T + 4M\right\} 4\varkappa^+\varkappa^-\pi^{-1}[2\varkappa + (\varkappa^+)^2]^{-1}, \quad \Gamma_1 = \Gamma(1/4), \quad \Gamma_0 = \Gamma(3/4)
\end{aligned}$$

and the following transcendental equation for determining the point b_2

$$\begin{aligned}
&C_1\theta_{11}(7/4) + C_2\theta_{21}(5/4) + \gamma_*[e_1\zeta_{11}(7/4) - e_2\zeta_{21}(5/4)] + \\
&+\sum_{k=1}^\infty \left[-\eta_{1k1}\left(\frac{7}{4}\right)A_{1k} - \eta_{2k1}\left(\frac{5}{4}\right)A_{2k} + \frac{A_{4k}}{k - 1/2}\right] + O\left\{\left(\frac{b_1}{b_2}\right)^{1/2}\right\} = 0, \quad b_1 \to 0
\end{aligned} \tag{2.12}$$

$$C_1=\gamma_* e_1-\frac{T}{2\Gamma_1}+\sum_{k=1}^{\infty}\frac{A_{1k}}{k-1/4},\quad C_2=-\gamma_* e_2+\frac{\varkappa T}{2\varkappa^+\Gamma_0}+\sum_{k=1}^{\infty}\frac{A_{2k}}{k-3/4}$$

$$\|\eta_{mkn},\theta_{mn},\zeta_{mn}\|(t)=\sum_{j=1}^{\infty}\frac{b_2^{j-9/4+m/2}h_{mj}}{n+j-t}\left\|\frac{1}{k+j-1},1,\frac{1}{j-9/4+m/2}\right\|$$

$$h_{1n}=-\frac{\varkappa^-\Gamma(n-3/4)|\Gamma(2n-1+i\beta)|^2}{\pi 2^{1/2}\Gamma(n)\Gamma(2n-3/2)\Gamma(2n-1)},\quad h_{2n}=-\frac{\varkappa^+\varkappa^-\Gamma(n-1/4)|\Gamma(2n+i\beta)|^2}{\pi\varkappa 2^{1/2}\Gamma(n)\Gamma(2n-1/2)\Gamma(2n)}$$

where $\beta=(2\pi)^{-1}\ln\varkappa$. As a consequence of the fact that $b_1 \ll b_2$ (the quantity b_1 is smaller than the length of the slippage zone found in Section 1 in the symmetric case), the coefficients A_{mk} satisfy the following infinite system

$$A_{mn}=\alpha_{mn}b_2^{n-\lambda_m+7/4}\left\{f_{mn}+\sum_{k=1}^{\infty}\left[E_{1kn}(\lambda_m)A_{1k}+E_{2kn}\left(\lambda_m-\frac{1}{2}\right)A_{2k}+\right.\right.$$

$$\left.\left.+(n+k-\lambda_m+1/4)^{-1}A_{4k}\right]\right\}\quad(m=1,2,3)$$

$$A_{4n}=\varkappa\alpha_{3n}b_2^{n-3/2}\left[f_{4n}+\sum_{k=1}^{\infty}\frac{(n-1/2)A_{3n}}{(n+k-1)(-k+1/2)}\right]\quad(n=1,2,\ldots)$$

where the following notation has been adopted

$$\lambda_1=1,\quad\lambda_2=3/2,\quad\lambda_3=5/4,\quad E_{jkn}(t)=-\eta_{jkn}(t)+(k-j/2+1/4)^{-1}\theta_{jn}(t)$$

$$f_{mn}=F_n(\lambda_m,\lambda_m-1/2)\quad(m=1,2,3);\quad f_{4n}=\gamma_* e_3(-n+1/2)(n-3/2)^{-1}-\pi^{-1/2}P$$

$$F_n(p,q)=1/2T[-\Gamma_1^{-1}\theta_{1n}(p)+(\varkappa/\varkappa^+)\Gamma_0^{-1}\theta_{2n}(q)]+$$

$$+\gamma_*\{e_1[\theta_{1n}(p)+\zeta_{1n}(p)]-e_2[\theta_{2n}(q)+\zeta_{2n}(q)]\}$$

$$\alpha_{mn}=\frac{\Gamma(n+7/4-\lambda_m)|\Gamma(2n+2-2\lambda_m+i\beta)|^2\mu_m}{\pi\Gamma(n)\Gamma(2n+2-2\lambda_m)\Gamma(2n+5/2-2\lambda_m)},\quad\mu_1=-\frac{\varkappa^-}{2\varkappa},\quad\mu_2=-\frac{\varkappa^-}{2\varkappa^+},\quad\mu_3=\frac{\varkappa^+}{\varkappa}$$

The length of the smaller slippage zone is sought from the condition for the profile of the slit to be smooth in the neighbourhood of the point $x = b_1$. Using asymptotic analysis and allowing for the smallness of the quantity b_1, we find

$$b_1=b_2\exp\left\{-\frac{1}{\beta}\operatorname{arctg}\frac{2R_1R_2}{R_1^2-R_2^2}-\frac{2\pi}{\beta}\left(m+\frac{1}{2}\right)\right\},\quad m=0,1,\ldots$$

$$R=R_1+iR_2=\frac{1}{4^{2i\beta}}\left\{F_1\left(2+\frac{i\beta}{2},\ \frac{3}{2}+\frac{i\beta}{2}\right)+\right.$$

$$\left.+\sum_{k=1}^{\infty}\left[E_{1k1}\left(2+\frac{i\beta}{2}\right)A_{1k}+E_{2k1}(3/2+i\beta/2)A_{2k}+(k-3/4-i\beta/2)^{-1}A_{4k}\right]\right\},\quad \operatorname{Im}\{R_1,R_2\}=0 \tag{2.13}$$

As in Section 1, from all of the lengths $b_1 \ll b_2$, we choose the longest, that is, the length which corresponds to the case when $m = 0$.

The solution of problems (2.8) and (2.11) for values of the quantities b_1 and b_2, which satisfy Eq. (2.12) and relation (2.13), generates a solution of system (2.2)–(2.5) possessing the properties

$$\chi_j(x)=O(x^{-3/4}),\quad x\to 0;\quad \chi_j(x)=O\{(1-x)^{-3/4}\},\quad x\to 1\quad(j=1,2,3)$$

$$\chi_4(x)=O\{(x-b_1)^{1/2}\},\quad x\to b_1;\quad \chi_4(x)=O\{(b_2-x)^{1/2}\},\quad x\to b_2$$

Numerical calculations were carried out for $\nu = 0.3$ and $T = 0$. We present the values of $P^{-1}b_2$ ($1 - b_2$—the length of the larger slippage zone) and $2G\gamma P^{-1}$ (γ is the turn angle of the inclusion) when $M = -Pe$ for several values of the eccentricity e (when $-1 \leqslant e < -1/2$ it is assumed that a pair of forces P and $-P$ with an eccentricity $e/2$ is applied to the inclusion)

e	−1	−0.8	−0.6	−0.5	−0.4	−0.3	−0.1
$P^{-1}b_2$	0.835	0.848	0.865	0.876	0.890	0.906	0.954
$2GP^{-1}\gamma$	−3.17	−2.76	−2.35	−2.15	−1.95	−1.74	−1.33

If, however, $P = 0$, then only the magnitude of the turn angle changes when there is a change in M and the lengths of the slippage zones for fixed ν do not vary. The greatest length of the right-hand slippage zone occurs when $\nu = 0$: $b_2 = 0.7438$. Values of the turn angle of the inclusion are presented below for some values of the moment M

M	−100	−10	−5	−2	−1	−0.5	−0.1
$2G\gamma$	−203	−20.3	−10.2	−4.07	−2.03	−1.02	−0.203

It can be seen that the turn angle is a linear function of the moment (when $P = 0$).

This research was carried out with support from the International Association for Cooperation and Collaboration with Scientists from the Independent States of the former Soviet Union (INTAS-93-2600).

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Translated by E.L.S.